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Chromatic coloring with a maximum color class

Bor-Liang Chen[a], Kuo-Ching Huang[b], Chih-Hung Yen[c,*]

[a] *Department of Business Administration, National Taichung Institute of Technology, Taichung 40401, Taiwan*
[b] *Department of Applied Mathematics, Providence University, Shalu, Taichung 43301, Taiwan*
[c] *Department of Applied Mathematics, National Chiayi University, Chiayi 60004, Taiwan*

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Abstract

Let G be any graph, and also let $\Delta(G)$, $\chi(G)$ and $\alpha(G)$ denote the *maximum degree*, the *chromatic number* and the *independence number* of G, respectively. A *chromatic coloring* of G is a proper coloring of G using $\chi(G)$ colors. A color class in a proper coloring of G is *maximum* if it has size $\alpha(G)$. In this paper, we prove that if a graph G (not necessarily connected) satisfies $\chi(G) \geq \Delta(G)$, then there exists a chromatic coloring of G in which some color class is maximum. This cannot be guaranteed if $\chi(G) < \Delta(G)$. We shall also give some other extensions.

Keywords: Vertex coloring; Chromatic coloring; Maximum degree; Chromatic number; Independence number

1. Introduction

Throughout this paper, all graphs considered are finite, undirected, loopless and without multiple edges. We refer the reader to [2] for terminology in graph theory. A *proper k-coloring* of a graph G is a labeling $f : V(G) \to \{1, 2, \ldots, k\}$ such that adjacent vertices have different labels. The labels are *colors*; the vertices of one color form a *color class*. The *chromatic number* of a graph G, written $\chi(G)$, is the least k such that G has a proper k-coloring. A *chromatic coloring* of a graph G is a proper coloring of G using $\chi(G)$ colors.

Furthermore, a *clique* in a graph is a set of pairwise adjacent vertices. In contrast to a clique, an *independent set* in a graph is a set of pairwise nonadjacent vertices. The *independence number* of a graph G, written $\alpha(G)$, is the maximum size of an independent set in G. An independent set in a graph G is *maximum* if it has size $\alpha(G)$.

Let $\Delta(G)$ denote the *maximum degree* of a graph G. In 1941, Brooks [1] proposed the following result.

Theorem 1.1. *If G is a connected graph other than an odd cycle or a complete graph, then $\chi(G) \leq \Delta(G)$.*

Therefore, a chromatic coloring of a connected graph G uses at most $\Delta(G) + 1$ colors. Moreover, since a color class is obviously an independent set, we would like to know whether there exists a chromatic coloring of G in which some color class is a maximum independent set, namely, some color class is *maximum*. However, this result cannot

* Corresponding author. Tel.: +886 5 2717873; fax: +886 3 5257917.
E-mail address: chyen@mail.ncyu.edu.tw (C.-H. Yen).

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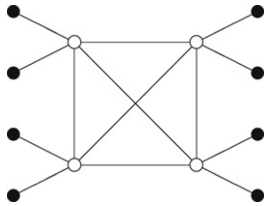

Fig. 1. A graph and the maximum independent set in it.

be guaranteed if $\chi(G) < \Delta(G)$. Let us consider a connected graph H consisting of a complete graph of order $n \geq 2$ in which each vertex is joined additionally to two distinct isolated vertices. The graph in Fig. 1 shows the case when $n = 4$. Since the maximum independent set in H is unique, say S, and $H - S$ is a complete graph K_n, we have $\chi(H - S) = \chi(H) = n < n + 1 = \Delta(H)$. Hence, every chromatic coloring of H cannot have a color class which is maximum.

The purpose of this paper is to prove that if a graph G (not necessarily connected) satisfies $\chi(G) \geq \Delta(G)$, then there exists a chromatic coloring of G in which some color class is maximum. In addition, we shall also give some other extensions.

2. Some preliminary results

Lemma 2.1. *Let G be a graph with $\chi(G) \geq \Delta(G)$, and also let S be a maximum independent set in G. Then $\chi(G - S) = \chi(G) - 1$ if and only if each component of $G - S$ is not an odd cycle when $\chi(G) = 3$ or a complete graph of order $\chi(G)$ when $\chi(G) \neq 3$.*

Proof. ($\Rightarrow$) It is trivial. ($\Leftarrow$) Suppose that $\chi(G - S) \neq \chi(G) - 1$. Then $\chi(G - S) = \chi(G)$. Hence, there must exist one component G_i of $G - S$ such that $\chi(G_i) = \chi(G)$. Since S is a maximum independent set in G, each vertex of $V(G) - S$ in G must be adjacent to some vertex of S, and $\Delta(G) > \Delta(G - S) \geq \Delta(G_i)$. Then, by $\chi(G_i) = \chi(G) \geq \Delta(G) > \Delta(G_i)$ and Theorem 1.1, we have that either G_i is an odd cycle with $\chi(G_i) = \chi(G) = 3$, or G_i is a complete graph with $|V(G_i)| = \chi(G_i) = \chi(G) \neq 3$. This is a contradiction. □

Before we go any further, some basic notions needed in the sequel are collected as follows. Let G be any graph. An *odd path-component* or an *odd cycle-component* of G is a component of G isomorphic to an odd path or an odd cycle. Similarly, a *K_n-component* of G is a component of G isomorphic to a complete graph of order n. Given a nonempty proper subset S of $V(G)$, a *$(S, \bar{S})$-chain* in G is a path that alternates between vertices in S and vertices in $\bar{S}$, where $\bar{S}$ denotes $V(G) - S$. Furthermore, the set consisting of the neighbors of vertices of S in G is denoted by $N_G(S)$. If a path P in G is from vertex u to vertex v, then u and v are the *endpoints* of P.

Lemma 2.2. *Let G be a connected graph with $\chi(G) = \Delta(G) = 3$. Then there exists a maximum independent set S in G such that $\chi(G - S) = 2$.*

Proof. By Lemma 2.1, it suffices to show that there exists a maximum independent set S in G such that $G-S$ contains no odd cycle-components. Hence, among all maximum independent sets in G, we let S be one satisfying that $G - S$ contains the least number of odd cycle-components, and denote such a number by t. We claim that $t = 0$.

Suppose otherwise. Then $t \geq 1$, and we use C to denote some odd cycle-component of $G - S$. Consider any vertex v_1 in C. Since S is a maximum independent set in G and $\Delta(G) = 3$, there must exist exactly a vertex x_1 of S adjacent to v_1 in G and $\Delta(G - S) = \Delta(G) - 1 = 2$. Now, let $P = v_1 - x_1 - v_2 - x_2 - \cdots$ be a maximal $(S, \bar{S})$-chain from v_1 in G. Then $v_i \in \bar{S}$ and $x_i \in S$ for all $i \geq 1$. Furthermore, let r denote the least i such that x_i has less than 3 neighbors in G or the two neighbors of x_i other than v_i in G are not exactly the two endpoints of some odd path-component of $G - S$.

If r does not exist, then $N_G(x_i) = \{v_i, u_{i+1}, v_{i+1}\}$ where u_{i+1} and v_{i+1} are exactly the two endpoints of some odd path-component of $G-S$ for each $i \geq 1$. Also, the endpoint of P other than v_1 must be a vertex of $\bar{S}$, denoted by v_{z+1}, where $z \geq 1$. Moreover, $\{v_1, v_2, \ldots, v_{z+1}\}$ is independent in G and $N_G(\{v_1, v_2, \ldots, v_{z+1}\}) \cap S = \{x_1, x_2, \ldots, x_z\}$. Fig. 2 shows such a case. Hence, we can let $S' = (S - \{x_1, x_2, \ldots, x_z\}) \cup \{v_1, v_2, \ldots, v_{z+1}\}$ be an independent set of size $|S| + 1$ in G. But this is a contradiction.

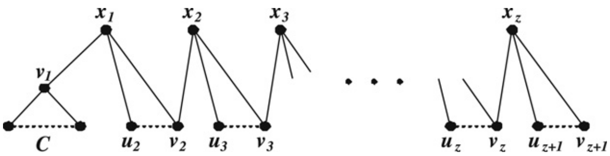

Fig. 2. The case if r does not exist.

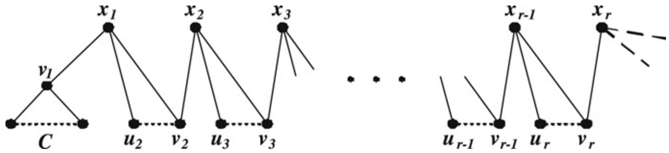

Fig. 3. The case when r exists.

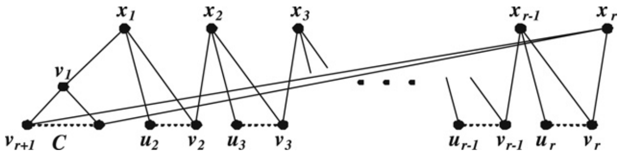

Fig. 4. The case when r exists and $N_G(x_r) - \{v_r\} = N_C(v_1)$.

Suppose that r exists. Then $N_G(x_i) = \{v_i, u_{i+1}, v_{i+1}\}$ where u_{i+1} and v_{i+1} are exactly the two endpoints of some odd path-component of $G - S$ for $1 \le i \le r - 1$. And, x_r has less than 3 neighbors in G or the two neighbors of x_r other than v_r in G are not exactly the two endpoints of some odd path-component of $G - S$. Also, $\{v_1, v_2, \dots, v_r\}$ is independent in G and $N_G(\{v_1, v_2, \dots, v_r\}) \cap S = \{x_1, x_2, \dots, x_r\}$ (see Fig. 3). Now, let $S' = (S - \{x_1, x_2, \dots, x_r\}) \cup \{v_1, v_2, \dots, v_r\}$. Then S' is also a maximum independent set in G and $\Delta(G - S') \le \Delta(G) - 1 = 2$. Since $\Delta(G - S') \le 2$, each of x_r's neighbors in $G - S'$ can only be a vertex of $N_{G-S}(\{v_1, v_2, \dots, v_r\})$ or a vertex of degree at most 1 in $G - S$ which is none of $u_2, v_2, u_3, v_3, \dots, u_r, v_r$. Moreover, in $G - S'$, C is destroyed and each of $x_1, x_2, \dots, x_{r-1}$ cannot be a part of an odd cycle-component. So, x_r must be a part of an odd cycle-component of $G - S'$; otherwise, $G - S'$ contains $t - 1$ odd cycle-components and this is a contradiction. Thus we conclude that $N_{G-S'}(x_r) = N_{G-S}(v_1)$ (or $N_G(x_r) - \{v_r\} = N_C(v_1)$; see Fig. 4). Then we can let $S'' = (S - \{x_r\}) \cup \{v_{r+1}\}$ be a maximum independent set in G such that $G - S''$ contains $t - 1$ odd cycle-components. But this is also a contradiction.

Therefore, the hypothesis $t \ge 1$ is wrong and $t = 0$. $\square$

Lemma 2.3. *Let G be a connected graph with $\chi(G) = \Delta(G) \ge 4$. Then there exists a maximum independent set S in G such that $\chi(G - S) = \chi(G) - 1$.*

Proof. By Lemma 2.1, it suffices to show that there exists a maximum independent set S in G such that $G - S$ contains no $K_{\chi(G)}$-components. Hence, among all maximum independent sets in G, we let S be one satisfying that $G - S$ contains the least number of $K_{\chi(G)}$-components, and denote such a number by t. We claim that $t = 0$.

Suppose otherwise. Then $t \ge 1$, and we use D to denote some $K_{\chi(G)}$-component of $G - S$. Consider any vertex v_1 in D. Since S is a maximum independent set in G and $\Delta(G) = \chi(G)$, there must exist exactly a vertex x_1 of S adjacent to v_1 in G and $\Delta(G - S) = \Delta(G) - 1 = \chi(G) - 1$. Now, let $P = v_1 - x_1 - v_2 - x_2 - \cdots$ be a maximal $(S, \bar{S})$-chain from v_1 in G. Then $v_i \in \bar{S}$ and $x_i \in S$ for all $i \ge 1$. Furthermore, let r denote the least i such that x_i has less than $\chi(G)$ neighbors in G or the $\chi(G) - 1$ neighbors of x_i other than v_i in G are not exactly the $\chi(G) - 1$ vertices of some $K_{\chi(G)-1}$-component of $G - S$.

If r does not exist, then $|N_G(x_i)| = \chi(G)$, $\{v_i, v_{i+1}\} \subseteq N_G(x_i)$ and the $\chi(G) - 1$ neighbors of x_i other than v_i in G are exactly the $\chi(G) - 1$ vertices of some $K_{\chi(G)-1}$-component of $G - S$ for each $i \ge 1$. Also, the endpoint of P other than v_1 must be a vertex of $\bar{S}$, denoted by v_{z+1}, where $z \ge 1$. Moreover, $\{v_1, v_2, \dots, v_{z+1}\}$ is independent in G and $N_G(\{v_1, v_2, \dots, v_{z+1}\}) \cap S = \{x_1, x_2, \dots, x_z\}$. Fig. 5 shows such a case when $\chi(G) = 4$. Hence, we can let $S' = (S - \{x_1, x_2, \dots, x_z\}) \cup \{v_1, v_2, \dots, v_{z+1}\}$ be an independent set of size $|S| + 1$ in G. But this is a contradiction.

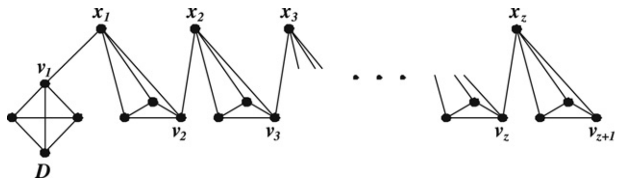

Fig. 5. The case if r does not exist and $\chi(G) = 4$.

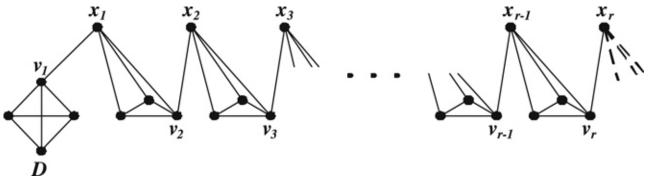

Fig. 6. The case when r exists and $\chi(G) = 4$.

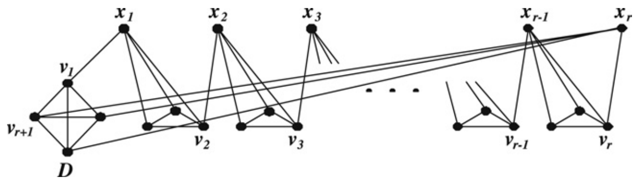

Fig. 7. The case when r exists, $\chi(G) = 4$ and $N_G(x_r) - \{v_r\} = N_D(v_1)$.

Suppose that r exists. Then $|N_G(x_i)| = \chi(G)$, $\{v_i, v_{i+1}\} \subseteq N_G(x_i)$ and the $\chi(G) - 1$ neighbors of x_i other than v_i in G are exactly the $\chi(G) - 1$ vertices of some $K_{\chi(G)-1}$-component of $G - S$ for $1 \le i \le r - 1$. And, x_r has less than $\chi(G)$ neighbors in G or the $\chi(G) - 1$ neighbors of x_r other than v_r in G are not exactly the $\chi(G) - 1$ vertices of some $K_{\chi(G)-1}$-component of $G - S$. Also, $\{v_1, v_2, \ldots, v_r\}$ is independent in G and $N_G(\{v_1, v_2, \ldots, v_r\}) \cap S = \{x_1, x_2, \ldots, x_r\}$. Fig. 6 shows such a case when $\chi(G) = 4$. Now, let $S' = (S - \{x_1, x_2, \ldots, x_r\}) \cup \{v_1, v_2, \ldots, v_r\}$. Then S' is also a maximum independent set in G and $\Delta(G-S') \le \Delta(G)-1 = \chi(G)-1$. Since $\Delta(G-S') \le \chi(G)-1$, each of x_r's neighbors in $G - S'$ can only be a vertex of $N_{G-S}(v_1)$ or a vertex of degree at most $\chi(G) - 2$ in $G - S$ which is none of $v_2, v_3, \ldots, v_r$. Moreover, in $G - S'$, D is destroyed and each of $x_1, x_2, \ldots, x_{r-1}$ cannot be a part of a $K_{\chi(G)}$-component. So, x_r must be a part of a $K_{\chi(G)}$-component of $G - S'$; otherwise, $G - S'$ contains $t - 1$ $K_{\chi(G)}$-components and this is a contradiction. Thus we conclude that $N_{G-S'}(x_r) = N_{G-S}(v_1)$ (or $N_G(x_r) - \{v_r\} = N_D(v_1)$). Fig. 7 shows such a case when $\chi(G) = 4$. Then we can let $S'' = (S - \{x_r\}) \cup \{v_{r+1}\}$ be a maximum independent set in G such that $G - S''$ contains $t - 1$ $K_{\chi(G)}$-components. But this is also a contradiction.

Therefore, the hypothesis $t \ge 1$ is wrong and $t = 0$. □

In fact, given any maximum independent set S in a connected graph G with $\chi(G) = \Delta(G)$, if $G - S$ contains $t \ge 1$ odd cycle-components when $\chi(G) = 3$ (or $K_{\chi(G)}$-components when $\chi(G) \ge 4$), then we always can find a new maximum independent set S' in G such that $G - S'$ contains $t - 1$ odd cycle-components (or $K_{\chi(G)}$-components).

3. The main results

We apply the results of the last section to the chromatic coloring with a maximum color class problem.

Theorem 3.1. *Let G be a graph with $\chi(G) \ge \Delta(G)$. Then there exists a maximum independent set S in G such that $\chi(G - S) = \chi(G) - 1$.*

Proof. Suppose that G consists of the components $G_1, G_2, \ldots, G_t$, where $t \ge 1$. It suffices to claim that there exists a maximum independent set S_i in each component G_i such that $\chi(G_i - S_i) \le \chi(G) - 1$.

First, if $\chi(G_i) \le \chi(G) - 1$, then any maximum independent set S_i in G_i has the property that $\chi(G_i - S_i) \le \chi(G_i) \le \chi(G) - 1$. Next, if $\chi(G_i) = \chi(G) > \Delta(G_i)$, then G_i is an odd cycle when $\chi(G_i) = 3$ or a complete graph when $\chi(G_i) \neq 3$ by Theorem 1.1. Moreover, if $\chi(G_i) = \chi(G) = \Delta(G_i) = 2$, then G_i is a path or an even cycle. In each of these two cases, it is not difficult to find a maximum independent set S_i in G_i such that $\chi(G_i - S_i) = \chi(G_i) - 1 = \chi(G) - 1$. Finally, if $\chi(G_i) = \chi(G) = \Delta(G_i) \geq 3$, then there exists a maximum independent set S_i in G_i such that $\chi(G_i - S_i) = \chi(G_i) - 1 = \chi(G) - 1$ by Lemmas 2.2 and 2.3. □

It is easy to see that a graph G has a chromatic coloring in which some color class is maximum if and only if there exists a maximum independent set S in G such that $\chi(G - S) = \chi(G) - 1$. Hence, by Theorem 3.1, we've reached what we wanted:

Theorem 3.2. *Let G be a graph with $\chi(G) \geq \Delta(G)$. Then there exists a chromatic coloring of G in which some color class is maximum.*

Now, let us use $\chi_{\max}(G)$ to denote the least k such that a graph G has a proper k-coloring in which some color class is maximum.

Proposition 3.3. $\chi(G) \leq \chi_{\max}(G) \leq \chi(G) + 1$ *for any graph G.*

Proof. Let S be a maximum independent set in G. Since $G - S$ is a subgraph of G, we have $\chi(G - S) \leq \chi(G)$. Then it is easy to obtain that $\chi_{\max}(G) \leq \chi(G - S) + 1 \leq \chi(G) + 1$ by adding the additional color class S to a chromatic coloring of $G - S$. Besides, it is trivial that $\chi_{\max}(G) \geq \chi(G)$. □

Corollary 3.4. *Let G be a graph with $\chi(G) \geq \Delta(G)$. Then $\chi_{\max}(G) = \chi(G)$.*

Proof. By Theorem 3.2 and Proposition 3.3. □

Corollary 3.5. *If G is a connected graph other than an odd cycle or a complete graph, then $\chi_{\max}(G) \leq \Delta(G)$.*

Proof. By Theorem 1.1, we have $\chi(G) \leq \Delta(G)$. If $\chi(G) \leq \Delta(G) - 1$, then $\chi_{\max}(G) \leq \chi(G) + 1 \leq (\Delta(G) - 1) + 1 = \Delta(G)$ by Proposition 3.3. If $\chi(G) = \Delta(G)$, then $\chi_{\max}(G) = \chi(G) = \Delta(G)$ by Corollary 3.4. Hence, the assertion holds. □

Corollary 3.5 implies that Brooks' Theorem (or Theorem 1.1) holds even if we require that the proper coloring has one color class which is a maximum independent set. Furthermore, Theorem 3.2 will also be used in [3] to prove the necessary and sufficient condition for a graph G (not necessarily connected) with $\Delta(G) = 3$ to be equitably $\Delta(G)$-colorable.

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